

Nonperturbative universal Chern-Simons theory

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Abstract. Closed simple integral representation through Vogel's universal parameters is found both for perturbative and nonperturbative parts of free energy of Chern-Simons theory on S^3 . This proves the universality of Chern-Simons partition function. For classical groups partition function manifestly satisfy $N \rightarrow -N$ duality, in apparent contradiction with previously used ones. For $SU(N)$ we show that asymptotic of nonperturbative part of our partition function coincides with that of Barnes G-function, recover Chern-Simons/topological string duality in genus expansion and resolve abovementioned contradiction. We discuss few possible directions of development of these results: derivation of representation of free energy through Gopakumar-Vafa invariants, possible appearance of non-perturbative additional terms, $1/N$ expansion for exceptional groups, duality between string coupling constant and Kähler parameters, etc.

1. INTRODUCTION

Many theories require for their definition Lie algebra, particularly simple Lie algebra. Due to well-known classification of these algebras this choice is considered as discrete one, although properties of theories with different simple Lie algebras often are expected to be similar. It was 't Hooft [1] who first promote the integer parameter N of $SU(N)$ gauge group into continuous one and suggest an extremely fruitful idea of $1/N$ expansion of gauge theories, connecting them with strings theory. This idea naturally extends to other classical groups - $SO(N)$ and $Sp(2N)$. Of course, it requires some continuation of theories from the domain of integer N to entire real line and complex plane. Even assuming analyticity of functions this extension is not unique, since there are a lot of analytical functions with zero values at integer points. Possible (actually used in [1]) definition is based on a fact that gauge theory perturbation terms can be presented as polynomial (or rational) functions of N , and in this form they are extended to an arbitrary N . Further, it appears that $SO(2N)$ and $Sp(2N)$ gauge theories are connected by $N \rightarrow -N$ transformation [2, 3] ($SU(N)$ is self-dual w.r.t. the $N \rightarrow -N$). So, in this way gauge theories with classical groups are promoted into domain of arbitrary N , and joined into two objects instead of four classical series. But gauge theories with exceptional groups remain completely out of this extension and unification, and the question whether they can be obtained in a similar way as a discrete points on some line (or other manifold), from which physical quantities depend smoothly, and whether that manifold can include classical groups, remains completely open until recently.

An important development happens in [4], on the basis of Vogel's notion of Universal Lie algebra [5, 6]. In these works Vogel sought the most general weight

system for knot theory diagrams. Any simple Lie algebra provides such a weight system, question was whether one can find more general one. It was proved that one-index contraction of two vertexes (vertex is a structure constants in case of Lie algebra), which can be considered as an operator acting on a symmetric square of corresponding space (adjoint representation) satisfies a third order equation. Then one can try to parametrize weight systems by three parameters, eigenvalues of that operator. According to appearance of these parameters, theory is symmetric w.r.t. permutations of them. Also an overall factor is inessential, since it corresponds to change in normalization of invariant bilinear form on algebra. So these three Vogel parameters α, β, γ belong to the Vogel's plane which is by definition a projective plane factorized w.r.t. all permutations of projective parameters. It was calculated in [5] what are specific points in that plane simple Lie algebras correspond to. They are given in Table 1. Parameter t there denotes the sum $t = \alpha + \beta + \gamma$, last column shows that with normalization of first column t becomes a dual Coxeter number h^\vee , this corresponds to so called minimal bilinear invariant form on the simple Lie algebra, when the square of long root(s) is 2.

TABLE 1. Vogel's parameters for simple Lie algebras

| Type | Lie algebra | α | β | γ | $t = h^\vee$ |
|-------|------------------------|----------|---------|----------|--------------|
| A_n | \mathfrak{sl}_{n+1} | -2 | 2 | $(n+1)$ | $n+1$ |
| B_n | \mathfrak{so}_{2n+1} | -2 | 4 | $2n-3$ | $2n-1$ |
| C_n | \mathfrak{sp}_{2n} | -2 | 1 | $n+2$ | $n+1$ |
| D_n | \mathfrak{so}_{2n} | -2 | 4 | $2n-4$ | $2n-2$ |
| G_2 | \mathfrak{g}_2 | -2 | $10/3$ | $8/3$ | 4 |
| F_4 | \mathfrak{f}_4 | -2 | 5 | 6 | 9 |
| E_6 | \mathfrak{e}_6 | -2 | 6 | 8 | 12 |
| E_7 | \mathfrak{e}_7 | -2 | 8 | 12 | 18 |
| E_8 | \mathfrak{e}_8 | -2 | 12 | 20 | 30 |

Next, many quantities can be expressed through Vogel universal parameters by smooth functions. Such quantities are called universal. For example, dimension of adjoint representation and its second Casimir are universal quantities, since they are given by

$$(1) \quad \dim \mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}$$

$$(2) \quad C_2 = 2t$$

Dimensions naturally are integers for parameters from Table 1, although not only for them. In [7, 8, 9, 10, 11] a lot of dimension formulae have been presented, covering many irreducible representations of simple Lie algebras. In [12] the universal formula is presented for eigenvalues of higher Casimirs on an adjoint representation. In ref [13] the classification of simple Lie algebras is recovered, in a reasonable sense, by requirement of regularity of universal character (4 below) and it is shown that simple Lie algebras (and few similar objects) are in one to one correspondence with solutions of certain Diophantine equations.

But it is not guaranteed, that any quantity can be expressed through universal parameters. For example the universal formula for dimension of fundamental

representations of classical groups is not known (although they are connected by $N \rightarrow -N$ duality). They are "not universal" in this sense, although it is not a theorem, and there is no clear definition of irrep being "not universal". It is not excluded that corresponding formulae exist, but are more complicated.

The $N \rightarrow -N$ duality is overlapping with universality. If some quantity is universal, then for classical groups it is $N \rightarrow -N$ dual: as is seen from Table 1 that duality corresponds to a $\alpha \leftrightarrow \beta$ transposition together with projective transformation. For that quantities universality is wider notion, since it includes an exceptional groups. From the other hand, some quantities, which are duality related (as mentioned dimensions of fundamental representations of classical groups) have no known universal representation.

The main question now is whether physical theories, e.g. gauge theories, "respect" Vogel's parameters, i.e. whether physical quantities can be expressed through these parameters by some smooth functions, which give answers for particular group at values from Table 1. The main achievement of [4] is that it is proved for Chern-Simons gauge theory on S^3 that few important quantities are universal. Particularly, perturbative part of its partition function is universal, i.e. can be expressed through universal parameters, more exactly, each term in perturbative expansion is a rational function of universal parameters. Universal are also central charge and unknot Wilson loop.

In present paper we have developed [4] in two directions. First, we present a closed universal integral representation for perturbative part F_2 of free energy.

$$(3) \quad F_2 = \int_0^\infty dx \frac{F(x/\delta)}{x(e^x - 1)}$$

Here $F(x)$ is a universal expression [4] for character of adjoint representation minus dimension of algebra, evaluated at point $x\rho$, ρ is a Weyl vector in roots space, i.e. half of sum of all positive roots of a given simple Lie algebra.

$$(4) \quad F(x) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(\frac{x\alpha}{4})} \frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})} \frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})} - \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}$$

$$(5) \quad \delta = \kappa + t$$

κ is a coefficient in front of Chern-Simons action (more detailed definitions see in Section 2), δ is (an inverse) effective coupling constant.

Second, it is argued that the same expression (with minus sign) with $\kappa = 0$ essentially gives a non-perturbative part of free energy (exact expression is given by (31),(32),(34)). This particularly means that non-perturbative part is universal, also, which completes the proof of universality of total partition function of Chern-Simons theory. Since non-perturbative part of partition function is essentially an (inverse) invariant volume of gauge group, it means that we have found a universal expression for invariant volume of group, see (32). Its logarithm is essentially (3) with $\kappa = 0$. Note the beautiful expression for $1/N$ expansion of group volume given in [16], where coefficients are Euler characteristics of moduli space of curves of genus g . The generalization of this expression is one of the aims of our search of universal expression for volume. Particularly, when expanded over universal parameters, universal volume will provide some refinement of abovementioned virtual Euler characteristics, the topological interpretation of which is very interesting question, with possible influence on string theory.

This representation of free energy of Chern-Simons theory apparently differs from known answers for classical groups, particularly $SU(N)$. For perturbative part it is more or less evident, by construction, that when expanded into a series over coupling constant (actually over $1/\delta \sim 1/(k+N)$) it will give the same answers, polynomials over N . But for non-perturbative part one cannot be sure at all, because we are dealing with complicated function of N , which essentially is product over n of $n!$, n running from 1 to $N-1$. Usually this is continued to arbitrary complex N as a Barnes function $G(1+N)$, see [14, 15, 16], but since it is not unique one may wonder what is the relation between these two functions: Barnes (left side of Eq.79) and our one (right side of Eq.79). Moreover, usual continuation violates $N \rightarrow -N$ duality (Barnes function $G(1+N)$ is not even with respect to N , recovering this duality was another our aim), which is one of the questions we had to resolve when asking for a universal expression for free energy, because, as mentioned above, $N \rightarrow -N$ duality is part of universality. Indeed, expressions we obtained for $SU(N)$ partition function's perturbative and non-perturbative parts (24), (35), (79) are explicitly $N \rightarrow -N$ invariant, so we have an explicitly different (from the usual one) expression for non-perturbative part of $SU(N)$ Chern-Simons.

To resolve this puzzle and to work out technique of handling the integral representations of free energy, we calculate a genus expansion of free energy of $SU(N)$ theory and compare with that of [15] (see [17] for a review). I.e. we re-derive in our language of integral representations the proof of duality of Chern-Simons on S^3 with closed topological strings, known as a Gopakumar-Vafa geometrical transition. We show that their genus expansions coincide, particularly genus expansion of (essentially, few details have to be taken into account) Barnes function coincides with genus expansion of our non-perturbative universal expression for an $SU(N)$ group volume. This establishes the correctness of our representation of free energy of Chern-Simons theory for $SU(N)$. Its advantage is $N \rightarrow -N$ duality for $SU(N)$ (and $SO(2N)/Sp(2N)$). This also is a strong argument in favor of universal expression for free energy. We also explain apparent contradiction described above in Section 5.

In Conclusion we discuss few topics for future development. Among them are expression of free energy through Gopakumar-Vafa invariants - how it can appear in the present approach, and what are additional non-perturbative terms; $1/N$ expansion for exceptional groups and refinement of topological invariants; possible duality between string coupling constant and Kähler parameters, and others.

2. PERTURBATIVE PARTITION FUNCTION OF CHERN-SIMONS THEORY

Partition function of Chern-Simons theory is formally given by functional integral

$$(6) \quad Z(M) = \int DA \exp \left(\frac{i\kappa}{4\pi} \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right)$$

Here "Tr" means an unnormalized invariant bilinear form on algebra, κ is an arbitrary number, which has to become a positive integer k when "Tr" is normalized to minimal invariant bilinear form on algebra, see [4]. So κ has to be rescaled simultaneously with universal parameters. The values of universal parameters, corresponding to minimal invariant bilinear form are given in Table 1, then κ becomes an integer, level k , parameter t becomes dual Coxeter number h^\vee .

Partition function of Chern-Simons theory on S^3 , with a specific choice of trivialization of tangent bundle, is [18];

$$(7) \quad Z = S_{00} = Vol(Q^\vee)^{-1} \delta^{-r/2} \prod_{\mu \in R_+} 2 \sin \frac{\pi(\mu, \rho)}{\delta}.$$

where S_{00} is an element of matrix of modular S-transformations, $R(R_+)$ is the set of roots (positive roots) of algebra, Q^\vee is coroot lattice, $Vol(Q^\vee)$ is a volume of fundamental domain of that lattice, $\delta = \kappa + t$, product is over all positive roots of a given algebra.

Let's rewrite it as the product $Z = Z_1 Z_2$, where

$$(8) \quad Z_1 = Vol(Q^\vee)^{-1} \delta^{-r/2} \prod_{\mu \in R_+} \frac{2\pi(\mu, \rho)}{\delta}$$

and

$$(9) \quad Z_2 = \prod_{\mu \in R_+} \sin \frac{\pi(\mu, \rho)}{\delta} / \frac{\pi(\mu, \rho)}{\delta}.$$

The first, non-perturbative factor Z_1 has a geometric meaning (cf. [18, 16, 4]):

$$(10) \quad Z_1 = \frac{(2\pi\delta^{-1/2})^{\dim \mathfrak{g}}}{Vol(G)},$$

where $Vol(G)$ is the volume of the corresponding compact simply connected group, with metric given by minimal bilinear invariant form.

This factor will be discussed in next Section, here we shall calculate Z_2 , the perturbative part of free energy. Using infinite product representation

$$(11) \quad \sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \left(\frac{x}{n}\right)^2\right)$$

and

$$(12) \quad \ln \frac{\sin \pi x}{\pi x} = \sum_{n=1}^{\infty} \ln \left(1 - \left(\frac{x}{n}\right)^2\right) = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \frac{x^{2m}}{n^{2m}}$$

we get for perturbative free energy:

$$(13) \quad F_2 = -\ln Z_2 = \sum_{\mu \in R_+} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\left(\frac{(\mu, \rho)}{\delta}\right)^{2m}}{n^{2m}} =$$

$$(14) \quad = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{2m} \frac{p_{2m}}{(n\delta)^{2m}}$$

where we introduced

$$p_k = \sum_{\mu \in R} (\mu, \rho)^k.$$

We have $p_k = 0$ for all odd k and $p_{2m} = 2 \sum_{\mu \in R_+} (\mu, \rho)^{2m}$. Up to now, calculations are standard. Next, consider following exponential generating function,

which evidently is character in adjoint representation, evaluated at point $x\rho$, minus dimension of algebra [4]:

$$F(x) = \sum_{k=1}^{\infty} \frac{p_k}{k!} x^k = \sum_{\mu \in R} (e^{x(\mu, \rho)} - 1).$$

It was expressed in terms of the Vogel's parameters in [4], answer is (4), which shows that p_k and hence perturbative partition function are universal quantities. Expression for p_2 is a homogeneous form of the so called Freudenthal-de Vries strange formula:

$$(15) \quad \sum_{\mu \in R_+} (\mu, \rho)^2 = \frac{t^2}{12} \dim \mathfrak{g},$$

expressions for other p_k can be called a generalized Freudenthal-de Vries (strange) relations.

Now we transform (13) further by Borel summation of m-series in it. Answer can be expressed through $F(x)$:

$$(16) \quad F_2 = \sum_{n=1}^{\infty} \int_0^{\infty} dx \frac{e^{-x}}{x} F\left(\frac{x}{n\delta}\right) = \sum_{n=1}^{\infty} \int_0^{\infty} dx \frac{e^{-xn\delta}}{x} F(x) =$$

$$(17) \quad \int_0^{\infty} dx \frac{e^{-x\delta}}{x(1 - e^{-x\delta})} F(x) = \int_0^{\infty} \frac{dx}{x} \frac{F(x/\delta)}{(e^x - 1)}$$

Integrals converge, since at large x $F(x/\delta)$ behaves as exponent of x with index $\frac{t+\alpha/2}{\delta}$, which is always less than 1 in physical semiplane and at $\kappa \geq 0$. Physical semiplane [13] of Vogel's plane is semiplane where parameters have different signs, e.g. $\alpha < 0, \beta > 0, \gamma > 0$. All simple Lie algebras belong to physical semiplane. At $x = 0$ integral converges due to $F(x) \sim x^2$ at small x .

The other derivation of (16), which requires certain integral representation of gamma-function, is the following. Recall representation of sinus through gamma-functions:

$$(18) \quad \frac{\sin(\pi x)}{\pi x} = \frac{1}{\Gamma(1-x)\Gamma(1+x)}$$

Then perturbative free energy is:

$$(19) \quad F_2 = - \sum_{\mu \in R_+} \ln\left(\frac{\sin(\pi(\rho, \mu)/\delta)}{\pi(\rho, \mu)/\delta}\right)$$

$$(20) \quad = \sum_{\mu \in R_+} \ln(\Gamma(1 - (\rho, \mu)/\delta)) + \ln(\Gamma(1 + (\rho, \mu)/\delta))$$

Next we use integral representation of gamma-function [19]:

$$(21) \quad \ln \Gamma(1+z) = \int_0^{\infty} dx e^{-x} \frac{e^{-zx} + z(1 - e^{-x}) - 1}{x(1 - e^{-x})}$$

to get

$$(22) \quad F_2 = \int_0^\infty dx \frac{e^{-x} \sum_{\mu \in R_+} \left(e^{x \frac{(\rho, \mu)}{\delta}} + e^{-x \frac{(\rho, \mu)}{\delta}} - 2 \right)}{x(1 - e^{-x})} =$$

$$(23) \quad \int_0^\infty dx \frac{e^{-x}}{x(1 - e^{-x})} F(x/\delta)$$

which coincides with above.

Universal perturbative partition function expression specified for SU(N) gives

$$(24) \quad \int_0^\infty dx \frac{e^{-x}}{x(1 - e^{-x})} \left(\frac{\sinh(x \frac{N+1}{2(k+N)}) \sinh(x \frac{N-1}{2(k+N)})}{\sinh^2(\frac{x}{2(k+N)})} - (N^2 - 1) \right) =$$

$$(25) \quad \int_0^\infty dx \frac{e^{-x}}{x(1 - e^{-x})} \left(\frac{\cosh(\frac{Nx}{k+N}) - \cosh(\frac{x}{k+N})}{2 \sinh^2(\frac{x}{2(k+N)})} - (N^2 - 1) \right) =$$

$$(26) \quad \int_0^\infty dx \frac{e^{-x}}{x(1 - e^{-x})} \left(\frac{\cosh(\frac{Nx}{k+N}) - 1}{2 \sinh^2(\frac{x}{2(k+N)})} - N^2 \right)$$

which is explicitly invariant w.r.t. the duality $N \rightarrow -N, k \rightarrow -k$.

For $SO(N)$ we have:

$$(27) \quad \int_0^\infty dx \frac{e^{-x}}{x(1 - e^{-x})} F(\frac{x}{\delta})$$

$$(28) \quad F(\frac{x}{\delta}) = \frac{\cosh(\frac{x(N-4)}{4(k+N-2)}) \sinh(\frac{x(N-1)}{2(k+N-2)}) \sinh(\frac{Nx}{4(k+N-2)})}{\sinh(\frac{x}{2(k+N-2)}) \sinh(\frac{x}{k+N-2})} - \frac{N(N-1)}{2}$$

Calculation for $Sp(2N)$ gives the same answer as for $SO(2N)$ with $N \rightarrow -N, k \rightarrow -2k$, as should be according to $N \rightarrow -N$ duality of Chern-Simons [4].

3. NON-PERTURBATIVE PART OF PARTITION FUNCTION

Partition function of Chern-Simons theory on S^3 , as defined in [18], is equal to 1 at $k=0$. This can be checked explicitly on exact expressions, e.g. for $SU(N)$, in general it follows from the fact that an S_{00} element of modular transformations matrix at $k=0$ is unity since there is no nontrivial unitary representation of corresponding affine algebra. So, nonperturbative part of partition function can be expressed through perturbative (see (31) below) and is universal since latter one is universal. The same is right about volume of group, see (32).

Partition function is product of non-perturbative Z_1 and perturbative Z_2 parts:

$$(29) \quad Z = Z_1 Z_2$$

Perturbative part depends on κ : $Z_2 = Z_2(\kappa)$, dependence of Z_1 on κ is given by power of δ (10):

$$(30) \quad Z_1 = Vol^{-1}((2\pi\delta^{-1/2})^{dim})$$

At $\kappa = 0$ one have $\delta = t$ and $Z = 1$, as discussed above, so

$$(31) \quad Z_1 = \frac{1}{Z_2(\kappa=0)} \left(\frac{t}{\delta} \right)^{\text{dim}/2}$$

$$(32) \quad Vol = Z_2(\kappa=0)(2\pi t^{-1/2})^{\text{dim}}$$

This gives a universal integral representation for volume function. We also give a direct derivation of this expression in [20]

So, partition function can be expressed as:

$$(33) \quad Z = \frac{Z_2(\kappa)}{Z_2(0)} \left(\frac{t}{\delta} \right)^{\text{dim}/2}$$

and final form of complete free energy is

$$(34) \quad F_1 + F_2 = (\text{dim}/2) \ln(\delta/t) + \int_0^\infty \frac{dx}{x} \frac{F(x/\delta) - F(x/t)}{(e^x - 1)}$$

Correspondingly, logarithm of main multiplier of volume of SU(N) is at $\kappa = 0$:

$$(35) \quad \int_0^\infty dx \frac{1}{x(e^x - 1)} \left(\frac{\cosh(x) - \cosh(x/(N))}{2 \sinh^2(\frac{x}{2N})} - (N^2 - 1) \right) =$$

$$(36) \quad \int_0^\infty dx \frac{1}{x(e^x - 1)} \left(\frac{\cosh(x) - 1}{2 \sinh^2(\frac{x}{2N})} - N^2 \right) =$$

$$(37) \quad \int_0^\infty dx \frac{1}{x(e^x - 1)} \left(\frac{\sinh^2(\frac{x}{2})}{\sinh^2(\frac{x}{2N})} - N^2 \right) =$$

$$(38) \quad \int_0^\infty \frac{dx}{x} \left(\frac{1 - e^{-x}}{4 \sinh^2(\frac{x}{2N})} - \frac{N^2}{e^x - 1} \right)$$

4. TOPOLOGICAL STRINGS AND 1/N EXPANSION OF SU(N) CHERN-SIMONS

Expressions for free energy, obtained in previous sections, in cases of classical groups should recover the known answers for a $1/\delta, 1/N$ series, which are shown to coincide with those for closed topological strings on a manifold with changed topology (Gopakumar-Vafa geometrical transition [15, 21, 22, 16]). We shall consider SU(N) case and obtain all terms in these expansions, which appear to coincide with known ones. We particularly present an integral representations for them, using a known integral representation for polylogarithm and other functions. This will be a non-trivial check for our formulae, particularly for universal expression for non-perturbative free energy of Chern-Simons theory.

With the use of identities

$$(39) \quad \partial \coth \frac{x}{2} = -\frac{1}{2 \sinh^2 \frac{x}{2}}$$

$$(40) \quad \cosh(x) = 2 \sinh^2(\frac{x}{2}) + 1$$

$$(41) \quad \coth \frac{x}{2} = \sum_{n=0}^{\infty} \frac{2B_{2n}x^{2n-1}}{(2n)!} = \frac{2}{x} + \frac{x}{6} - \frac{x^3}{8 \cdot 45} + \dots$$

one can get an expansion of $F(x)$ for $SU(N)$ over x :

$$(42) \quad F(x) = \left(\frac{\cosh(Nx) - \cosh(x)}{2 \sinh^2(\frac{x}{2})} - (N^2 - 1) \right) =$$

$$(43) \quad -1 + \partial \coth \frac{x}{2} - \cosh(Nx) \partial \coth \frac{x}{2} - (N^2 - 1) =$$

$$(44) \quad - \sum_{g=0, l=1}^{\infty} \frac{2(2g-1)}{(2g)!(2l)!} B_{2g} N^{2l} x^{2l+2g-2} - N^2$$

To get elements of perturbative expansion of free energy, we have to multiply each term on $(2l + 2g - 2)!/2(l + g - 1)$, substitute x by $1/(\delta n)$ and sum over n . All this happens inside our integral representation of free energy, if we insert in it the expansion of $F(x)$. To get a complete answer, with the purpose of establish a duality with topological strings, one have to add a terms from non-perturbative part, expanded in $1/N$. Below we shall recover the known proof of duality and show how it reappear in our language of integral representations of different contributions into free energy. These are the calculations of Gopakumar-Vafa [15], we shall compare with presentation of Mariño [17].

So, to make a contact with known results we multiply each term of (44) on $(2l + 2g - 2)!/2(l + g - 1)$, substitute x by $1/(\delta n)$ and sum over n . Also, introduce string coupling constant g_s and 't Hooft coupling (Kähler parameter) μ :

$$(45) \quad g_s = \frac{2\pi}{k + N}, \mu = \frac{2\pi i N}{k + N} = ig_s N$$

Then for $g \geq 2$:

$$(46) \quad \sum_{n=1}^{\infty} \frac{2(2g-1)}{(2g)!(2l)!} B_{2g} N^{2l} \left(\frac{1}{\delta n} \right)^{2l+2g-2} \frac{(2l+2g-2)!}{2(l+g-1)} =$$

$$(47) \quad - \frac{2B_{2g}\zeta(2l+2g-2)}{2g(2g-2)} C_{2l}^{2l+2g-3} \frac{N^{2l}}{\delta^{2l+2g-2}} =$$

$$(48) \quad - \frac{2B_{2g}\zeta(2l+2g-2)}{2g(2g-2)(2\pi)^{2g+2l-2}} C_{2l}^{2l+2g-3} N^{2l} (g_s)^{2l+2g-2}$$

which coincides essentially with (3.16), (3.18) of [17]. The same is true for terms with $g=0,1$, they coincide with (3.17) of [17].

Going back to our integral representation, insert into that of F_2 the expansion of character 42, and consider particular terms with fixed $g \geq 2$:

$$(49) \quad \int_0^\infty dx \frac{e^{-x}}{x(1-e^{-x})} (-1) \sum_{l=1}^{\infty} \frac{2(2g-1)}{(2g)!(2l)!} B_{2g} N^{2l} \frac{x^{2l+2g-2}}{\delta^{2l+2g-2}} =$$

$$(50) \quad \int_0^\infty dx \frac{e^{-x}}{x(1-e^{-x})} (-1) \frac{2(2g-1)B_{2g}}{(2g)!} \left(\frac{x}{\delta} \right)^{2g-2} \left(\cosh \frac{Nx}{\delta} - 1 \right)$$

Take one of two terms of last expression, namely last term, -1, in the last bracket:

$$(51) \quad \int_0^\infty dx \frac{e^{-x}}{x(1-e^{-x})} (-1) \frac{2(2g-1)B_{2g}}{(2g)!} \left(\frac{x}{\delta}\right)^{2g-2} (-1) =$$

$$(52) \quad \left(\frac{1}{\delta}\right)^{2g-2} \frac{2(2g-1)B_{2g}}{(2g)!} \int_0^\infty dx x^{2g-3} (e^{-x} + e^{-2x} + e^{-3} + \dots) =$$

$$(53) \quad \left(\frac{1}{\delta}\right)^{2g-2} \frac{2(2g-1)B_{2g}}{(2g)!} (2g-3)!(1 + \frac{1}{2^{2g-2}} + \frac{1}{3^{2g-2}} + \dots) =$$

$$(54) \quad \left(\frac{1}{\delta}\right)^{2g-2} \frac{2(2g-1)B_{2g}}{(2g)!} (2g-3)! \zeta(2g-2) =$$

$$(55) \quad \left(\frac{1}{\delta}\right)^{2g-2} \frac{2B_{2g}}{(2g-2)(2g)} (-1)^g \frac{B_{2g-2}(2\pi)^{2g-2}}{2(2g-2)!} =$$

$$(56) \quad g_s^{2g-2} \frac{(-1)^g B_{2g} B_{2g-2}}{(2g-2)(2g)(2g-2)!}$$

where we use

$$(57) \quad \zeta(2m) = (-1)^{m+1} \frac{B_{2m}(2\pi)^{2m}}{2(2m)!}$$

This is exactly the first terms in (5.28) and (5.32) of [17], up to sign, which comes from different definitions of free energy. It remains to obtain the last term in (5.32), i.e. that with polylogarithms. In [17] it is obtained as a sum of remaining perturbative terms in (5.28), which corresponds to term with \cosh above, and terms from non-perturbative Z_1 part above. So, we have to sum up that \cosh term and term of order N^{2-2g} from asymptotic expansion of group volume (more exactly, Z_1) term. Let's find that asymptotic expansion, later on we shall compare that with the expansion of Barnes G-function [23, 17, 16].

Nonperturbative contribution to free energy is

$$(58) \quad F_1 = -\ln Z_1 = \frac{1}{2}(N^2 - 1) \ln \frac{k+N}{N} - F_2(k=0)$$

$F_2(k=0)$ is:

$$(59) \quad F_2(k=0) = \int_0^\infty \frac{dx}{x} \left(\frac{1-e^{-x}}{4 \sinh^2(\frac{x}{2N})} - \frac{N^2}{e^x - 1} \right)$$

Integral is convergent, but to consider different contributions separately, let's regularize it at low limit by small ϵ , calculate contributions up to $O(\epsilon)$, then sum all contributions and send $\epsilon \rightarrow 0$. First contribution is:

$$(60) \quad \int_\epsilon^\infty \frac{dx}{x} \frac{1}{4 \sinh^2(\frac{x}{2N})} = \int_{\epsilon/N}^\infty \frac{dx}{x} \frac{1}{4 \sinh^2(\frac{x}{2})} = \frac{N^2}{2\epsilon^2} + \frac{1}{12} \ln \frac{\epsilon}{N} + c_1 + O(\epsilon)$$

Dependence on N of this integral completely comes from its dependence on ϵ , which is easily established by differentiation on ϵ and expansion function under integral around $x = 0$. Similar calculation for last contribution gives [24]:

$$(61) \quad \int_\epsilon^\infty \frac{dx}{x} \left(-\frac{N^2}{e^x - 1} \right) = -N^2 \left(\frac{1}{\epsilon} + \frac{1}{2} \ln \epsilon + \frac{1}{2} (\gamma - \ln(2\pi)) + O(\epsilon) \right)$$

The second contribution is:

$$(62) \quad \int_{\epsilon}^{\infty} \frac{dx}{x} \frac{-e^{-x}}{4 \sinh^2(\frac{x}{2N})}$$

Insert an expansion of $1/4 \sinh^2(\frac{x}{2N})$:

$$(63) \quad \int_{\epsilon}^{\infty} \frac{dx}{x} \frac{-e^{-x}}{4 \sinh^2(\frac{x}{2N})} = \int_{\epsilon}^{\infty} \frac{dx}{x} e^{-x} \left(\frac{N}{2} \partial \coth(\frac{x}{2N}) \right) =$$

$$(64) \quad \int_{\epsilon}^{\infty} \frac{dx}{x} e^{-x} \left(\frac{N}{2} \sum_{g=0}^{\infty} \frac{2B_{2g}(2g-1)}{(2g)!} \frac{x^{2g-2}}{N^{2g-1}} \right) =$$

$$(65) \quad -N^2 \int_{\epsilon}^{\infty} dx \frac{e^{-x}}{x^3} + \int_{\epsilon}^{\infty} dx \frac{e^{-x}}{12x} + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g} + O(\epsilon)$$

Evidently, for $g \geq 2$ asymptotic coincides with that of Barnes G-function, see (3.19) in [17]. For first two terms we have [24]:

$$(66) \quad \int_{\epsilon}^{\infty} \frac{dx}{x^3} e^{-x} = \frac{1}{2\epsilon^2} - \frac{1}{\epsilon} - \frac{1}{2} \ln \epsilon + \frac{3}{4} - \frac{1}{2} \gamma + O(\epsilon)$$

$$(67) \quad \int_{\epsilon}^{\infty} \frac{dx}{x} e^{-x} = -\ln \epsilon - \gamma + O(\epsilon)$$

Summing up all contributions, we see that singular at $\epsilon \rightarrow 0$ terms cancel, other terms give asymptotic for F_1 :

$$(68) \quad F_1 = -\ln Z_1 = \frac{1}{2} (N^2 - 1) \ln \frac{k+N}{N} - F_2(k=0) = \\ \frac{1}{2} (N^2 - 1) \ln \frac{k+N}{N} + \frac{1}{12} \ln N + \frac{3}{4} N^2 - \frac{1}{2} N^2 \ln(2\pi) - c_1 + \frac{1}{12} \gamma - \\ \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g}$$

This coincides exactly with non-perturbative contribution given in (3.19) of [17] provided (taking into account $\zeta'(-1) = 1/12 - \ln A$)

$$(69) \quad c_1 = \frac{1}{12} (\gamma + 1) - \ln A$$

Here γ, A are Euler-Mascheroni and Glaisher constants, respectively. We check this equality up to sixth digit by numerical calculation of integral with "Mathematica", so we believe it is correct. One also have to take into account that (3.19) of [17] includes a free energy of U(1) theory, taken there to be $-(1/2) \ln(N/(k+N))$, so one have to add same quantity to our SU(N) function (68) to compare to (3.19).

So, for $g \geq 2$ we sum up two contributions into free energy, perturbative and non-perturbative:

$$(70) \quad \begin{aligned} & \frac{2(2g-1)B_{2g}}{(2g)!(-1)} \int_0^\infty dx \left(\frac{e^{-x}}{x(1-e^{-x})} \left(\frac{x}{\delta} \right)^{2g-2} \cosh \frac{Nx}{\delta} + \frac{1}{2x} \left(\frac{x}{\delta} \right)^{2g-2} e^{-\frac{xN}{\delta}} \right) = \\ & \frac{2(2g-1)B_{2g}}{(2g)!(-1)} \int_0^\infty dx \frac{e^{-x/2}}{x(1-e^{-x})} \left(\frac{x}{\delta} \right)^{2g-2} (e^{x(-\frac{1}{2}+\frac{N}{\delta})} + e^{x(+\frac{1}{2}-\frac{N}{\delta})}) = \\ & \frac{2(2g-1)B_{2g}}{(2g)!(-1)} g_s^{2g-2} \int_0^\infty dx \frac{e^{-x/2}}{x(1-e^{-x})} \left(\frac{x}{2\pi} \right)^{2g-2} (e^{x(-\frac{1}{2}+\frac{\mu}{2\pi i})} + e^{x(+\frac{1}{2}-\frac{\mu}{2\pi i})}) \end{aligned}$$

and change integration variable $x \rightarrow 2\pi x$:

$$(71) \quad \frac{2(2g-1)B_{2g}}{(2g)!(-1)} g_s^{2g-2} \int_0^\infty dx \frac{x^{2g-3}}{\sinh(\pi x)} \cosh(x(\pi - \frac{\mu}{i}))$$

Compare with the following integral representation of polylogarithm [25]:

$$(72) \quad Li_s(z) = \int_0^\infty dx x^{-s} \frac{\sin(\frac{s\pi}{2} - x \ln(-z))}{\sinh(\pi x)}$$

For $s = 3 - 2g, g \geq 2$ integer

$$(73) \quad Li_{3-2g}(z) = \int_0^\infty dx x^{2g-3} \frac{\cos(x \ln(-z))}{\sinh(\pi x)}$$

$$(74) \quad Li_{3-2g}(e^{-\mu}) = \int_0^\infty dx x^{2g-3} \frac{\cos(x(i\pi - \mu))}{\sinh(\pi x)} =$$

$$(75) \quad \int_0^\infty dx x^{2g-3} \frac{\cosh(x(\pi - \frac{\mu}{i}))}{\sinh(\pi x)}$$

Comparing this with (71) we see that at $g \geq 2$ contribution into total free energy is

$$(76) \quad \frac{2(2g-1)B_{2g}}{(2g)!(-1)} g_s^{2g-2} Li_{3-2g}(e^{-\mu})$$

in agreement with (5.32) of [17].

5. BARNES FUNCTION AND GROUP VOLUME FUNCTION

As shown in previous Section, for SU(N) group the asymptotics of our non-perturbative part of Chern-Simons theory coincides with previously known, essentially that of Barnes G-function. From the other side, these two functions are really different, since our one is even with respect to $N \rightarrow -N$ duality, but G function manifestly not. Now we shall suggest an explanation of this apparent contradiction. Consider integral representation of F_1 :

$$(77) \quad F_1 = \frac{1}{2}(N^2 - 1) \ln \frac{k+N}{N} - \int_0^\infty \frac{dx}{x} \left(\frac{1 - e^{-x}}{4 \sinh^2(\frac{x}{2N})} - \frac{N^2}{e^x - 1} \right)$$

This integral exist at all complex N with non-zero real part. Let's try to connect points N and -N by some path in complex N plane such that integral in F_1 make sense on all points of that path. It is easy to see that there is no such a path, because

it inevitably has to pass through imaginary N axis, but integral is divergent at any point on that axis. From the other hand, any two points in semiplane with $\Re N > 0$, or any two points with $\Re N < 0$ can be connected by such a non-singular path. So we assume that our representation for non-perturbative free energy, and consequently for group volume defines two analytical functions of N , one for positive $\Re N$, and one for negative $\Re N$. $1/N$ expansions of these analytical functions coincide with $G(1+N)$ and $G(1-N)$ in positive and negative $\Re N$ semiplanes, respectively. The toy model for such a behavior is an integral

$$(78) \quad \int_0^\infty \frac{dx}{\cosh(zx)} = \frac{\pi}{2z}, \Re z > 0$$

which has features similar to our integral. It is not defined at $\Re z = 0$, as our one for $\Re N = 0$, it is symmetric w.r.t. the change $z \rightarrow -z$, points $z, -z$ cannot be connected by continuous path with regular values of integral due to singular line $\Re z = 0$. Calculation of integral gives an exact answer which actually provide an analytic continuation on all points on singular line except point $z=0$, which is true singularity. And in agreement with our assumption integral defines two analytic functions of z at $\Re z > 0$ and $\Re z < 0$: $\pi/2z$ and $-\pi/2z$, connected by $z \rightarrow -z$.

We assume that at positive $\Re N$ our function, having the same $1/N$ asymptotic expansion and the same value at integer positive points (by construction) coincides essentially with $G(1+N)$. Exactly, this means that we get (perhaps new form of) integral representation of logarithm of G -function (see [14, 16, 17] for connection of G and volume, for which we have an expression in above):

$$(79) \quad \ln(G(1+N)) = \frac{1}{2}N^2 \ln(N) - \frac{1}{2}(N^2 - N) \ln(2\pi) + \int_0^\infty \frac{dx}{x} \left(\frac{1 - e^{-x}}{4 \sinh^2(\frac{x}{2N})} - \frac{N^2}{e^x - 1} \right)$$

Taking asymptotic expansion of integral from (68):

$$(80) \quad \begin{aligned} \int_0^\infty \frac{dx}{x} \left(\frac{1 - e^{-x}}{4 \sinh^2(\frac{x}{2N})} - \frac{N^2}{e^x - 1} \right) = \\ -\frac{1}{12} \ln N - \frac{3}{4}N^2 + \frac{1}{2}N^2 \ln(2\pi) + c_1 - \frac{1}{12}\gamma + \\ \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g} \end{aligned}$$

we get an asymptotic expansion, which exactly coincides with that of $G(1+N)$, see e.g. [23, 22, 16]:

$$(81) \quad \ln(G(1+N)) = \frac{1}{2}N^2 \ln(N) - \frac{1}{2}(N^2 - N) \ln(2\pi) + \\ - \frac{1}{12} \ln N - \frac{3}{4}N^2 + \frac{1}{2}N^2 \ln(2\pi) + c_1 - \frac{1}{12}\gamma + \\ \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g} =$$

$$(82) \quad \left(\frac{1}{2}N^2 - \frac{1}{12}\right) \ln(N) + \frac{1}{2}N \ln(2\pi) - \frac{3}{4}N^2 + \frac{1}{12} - \ln A + \\ \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g}$$

This is the same coincidence which we observed earlier, comparing $1/N$ expansion of Chern-Simons and topological strings (68).

So, for general point in Vogel's plane our volume function is natural group-theory based generalization of Barnes G-function. For general point the analog of $N \rightarrow -N$ transformation is an exchange of parameters with different signs.

6. CONCLUSION: POSSIBLE DIRECTIONS OF DEVELOPMENT

In this Section we shall list possible directions of development of present results.

In their work [22] Gopakumar and Vafa derive a structure result for free energy of topological string through so called Gopakumar-Vafa (integer) invariants. For the simplest case it corresponds to $SU(N)$ Chern-Simons theory on S^3 and is equal to

$$(83) \quad \sum_{m=1}^{\infty} \frac{-e^{-m\mu}}{4m \sin^2(\frac{m\mu}{2})}$$

plus μ -independent terms given by (56). We shall recover below this result and show that actually some non-perturbative over string coupling constant terms are present.

Free energy is

$$(84) \quad F_1 + F_2 = (\dim/2) \ln(\delta/t) + \int_0^\infty \frac{dx}{x} \frac{F(x/\delta) - F(x/t)}{(e^x - 1)}$$

Let's extend integration to entire real x axis. Additional part is

$$(85) \quad \int_{-\infty}^0 \frac{dx}{x} \frac{F(x/\delta) - F(x/t)}{(e^x - 1)} = \int_0^\infty \frac{dx}{x} \frac{F(x/\delta) - F(x/t)}{(1 - e^{-x})} =$$

$$(86) \quad \int_0^\infty \frac{dx}{x} \frac{e^x(F(x/\delta) - F(x/t))}{(e^x - 1)} =$$

$$(87) \quad \int_0^\infty \frac{dx}{x} \frac{e^x}{(e^x - 1)} \left(\frac{\cosh(\frac{Nx}{k+N}) - 1}{2 \sinh^2(\frac{x}{2(k+N)})} - \frac{\cosh(x) - 1}{2 \sinh^2(\frac{x}{2N})} \right)$$

and is finite provided $|N| < 1$.

Difference is:

$$(88) \quad \int_0^\infty \frac{dx}{x} (F(x/\delta) - F(x/t))$$

This will be zero would integrals exist separately. But they diverge at upper limit Λ as

$$(89) \quad \int_0^\Lambda \frac{dx}{x} F(x) = -N^2 \ln \Lambda + c + O(1/\Lambda)$$

so difference is

$$(90) \quad N^2 (\ln \delta/t)$$

so

$$(91) \quad \int_{-\infty}^\infty = 2 \int_0^\infty + N^2 (\ln \delta/t)$$

$$(92) \quad \int_0^\infty = \frac{1}{2} \int_{-\infty}^\infty - (N^2/2) (\ln \delta/t)$$

So last term almost cancels in total free energy, remaining part is an U(1) free energy:

$$(93) \quad F_1 + F_2 = \int_{-\infty}^\infty \frac{dx}{x} \frac{1}{(e^x - 1)} \left(\frac{\sinh^2(\frac{Nx}{2(k+N)})}{\sinh^2(\frac{x}{2(k+N)})} - \frac{\sinh^2(\frac{x}{2})}{\sinh^2(\frac{x}{2N})} \right) - (1/2)(\ln \delta/t)$$

As mentioned, this expression is applicable at $|N| < 1$, so further results in principle are restricted to this (non-physical) domain. But it may be that further transformations will enlarge this domain of applicability, as happens in analytic continuation procedures, in all cases they deserve more careful further study.

We would like to close contour of integration by upper semicircle, for further shrinking it to the poles in upper semiplane. It seems that it is possible, due to analog of Jourdan's lemma. But we shall not discuss that issue now, since it evidently requires separate careful study, and simply assume that it is true, at least for some range of parameters, and look what will be the consequences.

In that case we can shrink contour to poles in upper semiplane and integral becomes the sum of residues in these poles. Poles are at points

$$(94) \quad x = 2\pi i p,$$

$$(95) \quad x = 2\pi i(k+N)n$$

$$(96) \quad x = 2\pi iNm$$

where p, n, m are integers to be chosen so that corresponding pole be in upper semiplane. For real positive k and N all positive p, n, m should be taken. Remind that there is no pole at $x=0$. Contribution of three series of poles are (assume there is no coincidence between them):

$$(97) \quad \sum_{m=1}^{\infty} \frac{-\cos(\frac{mN2\pi}{\delta}) + 1}{2m \sin^2(\frac{m\pi}{\delta})} = \sum_{m=1}^{\infty} \frac{1 - \cosh(m\mu)}{2m \sin^2(\frac{mg_s}{2})}$$

$$(98) \quad \sum_{m=1}^{\infty} i \frac{-1 + e^{-2\pi i N m} (1 + 2\pi i N m)}{2m^2\pi}$$

$$(99) \quad \sum_{m=1}^{\infty} \left(\frac{mN\pi \sin(2mN\pi) - \sin^2(mN\pi)}{m^2\pi^2 (e^{2mi\delta\pi} - 1)} - 2i\delta \frac{\sin^2(mN\pi)e^{2mi\pi\delta}}{m\pi (e^{2mi\delta\pi} - 1)^2} \right)$$

Actually (97) coincides with (83) plus (56), taking into account the invariance of given polylogarithm $Li_{3-2g}(e^{-\mu}) = Li_{3-2g}(e^\mu)$ in (76). Next contributions are non-perturbative w.r.t. the string coupling constant $g_s = 2\pi/\delta$. They have to be compared with existing non-perturbative calculations, see particularly [26].

Evidently, these calculations have to be clarified and justified in many respects, we hope to do that elsewhere.

Universal formulation of Chern-Simons theory open the way for the generalizations of many faces of that theory. One possibility is an extension of 1/N expansion on exceptional groups. In some sense it is already achieved. Indeed, in Vogel's plane all five exceptional groups are located on the line (usually denoted Exc) $\gamma = 2(\alpha + \beta)$, just as e.g. $SU(N)$ are located on line $\alpha + \beta = 0$. One can introduce any linear parameter on the line Exc , say $z = \alpha/\beta$ (for $SU(N)$ standard parameter is $N = 2\gamma/\beta$) and expand all universal quantities (partition function, adjoint Wilson loop, etc.) in a (Laurent) series w.r.t. the parameter z or z^{-1} . But of course it is not enough, we would like to have a topological interpretation of corresponding coefficients, just as for $SU(N)$ (and SO/Sp) Ooguri and Vafa [16] interpret coefficients in 1/N expansion as string theory objects, namely as a (virtual) Euler characteristics of moduli space of surfaces of a given genus, which are proportional to Bernoulli numbers, (68). In Exc line case, as in Section 4 for $SU(N)$, to get an expansion of free energy over parameter z one should substitute into expression for universal character (4) universal parameters as a (linear) functions of z and expand that over z . Difference appear first in that this object is not polynomial, or more exactly it becomes polynomial when common multiplier $\dim \mathfrak{g}$ (dimension of algebra) is separated. Second, corresponding expansion includes not Bernoulli numbers, but similar object relevant for Barnes' multiple gamma-functions [27].

Finally, this view from universality side suggests some duality between topological string's coupling constant g_s and Kähler parameter(s) μ . Indeed, initial universal functions (for free energy, e.g.) are symmetric w.r.t. the universal parameters. When restricted to $SU(N)$ some of these parameters become string coupling constant, namely $\alpha/\delta \sim g_s, \beta/\delta \sim g_s$, third one becomes Kähler parameter: $\gamma/\delta \sim \mu$, which naturally hints on some hidden duality between these parameters of topological string.

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